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PART II: MATHEMATICS

SECTION–I

Single Correct Choice Type

This section contains 8 multiple choice questions. Each question has 4 choices (A), (B), (C) and (D) for its answer, out of which **ONLY ONE** is correct.

21. Let z = x + iy be a complex number where x and y are integers. Then the area of the rectangle whose vertices are the roots of the equation $\bar{z}z^3 + z\bar{z}^3 = 350$ is

(A) 48 (B) 32

(C) 40 (D) 80

Sol. **(A)**

$z\bar{z}(\bar{z}^2 + z^2) = 350$

Put $z = x + iy$

$(x^2 + y^2)(x^2 - y^2) = 175$

$(x^2 + y^2)(x^2 - y^2) = 5 \cdot 5 \cdot 7$

$x^2 + y^2 = 25$

$x^2 - y^2 = 7$

$x = \pm 4, y = \pm 3$

$x, y \in I$

Area = 8 × 6 = 48 sq. unit.

22. If $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ are unit vectors such that $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = 1$ and $\vec{a}\cdot\vec{c} = \frac{1}{2}$, then

(A) $\vec{a}$, $\vec{b}$, $\vec{c}$ are non-coplanar (B) $\vec{b}$, $\vec{c}$, $\vec{d}$ are non-coplanar

(C) $\vec{b}$, $\vec{d}$ are non-parallel (D) $\vec{a}$, $\vec{d}$ are parallel and $\vec{b}$, $\vec{c}$ are parallel

Sol. **(C)**

$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = 1$ possible only when $|\vec{a}\times\vec{b}| = |\vec{c}\times\vec{d}| = 1$

and $(\vec{a}\times\vec{b}) \parallel (\vec{c}\times\vec{d})$

Since $\vec{a}\cdot\vec{c} = 1/2$ and $\vec{b} \parallel \vec{d}$, then $|\vec{c}\times\vec{d}| \neq 1$.

23. The line passing through the extremity A of the major axis and extremity B of the minor axis of the ellipse $x^2 + 9y^2 = 9$ meets its auxiliary circle at the point M. Then the area of the triangle with vertices at A, M and the origin O is

(A) $\frac{31}{10}$ (B) $\frac{29}{10}$

(C) $\frac{21}{10}$ (D) $\frac{27}{10}$

Sol. (D)

Equation of line AM is $x + 3y - 3 = 0$

Perpendicular distance of line from origin $= \frac{3}{\sqrt{10}}$

Length of AM $= 2\sqrt{9 - \frac{9}{10}} = 2 \times \frac{9}{\sqrt{10}}$

$\Rightarrow$ Area $= \frac{1}{2} \times 2 \times \frac{9}{\sqrt{10}} \times \frac{3}{\sqrt{10}} = \frac{27}{10}$ sq. units

24. Let $z = \cos\theta + i\sin\theta$. Then the value of $\sum_{m=1}^{15} \text{Im}\left(z^{2m-1}\right)$ at $\theta = 2^\circ$ is

(A) $\frac{1}{\sin 2^\circ}$ (B) $\frac{1}{3\sin 2^\circ}$

(C) $\frac{1}{2\sin 2^\circ}$ (D) $\frac{1}{4\sin 2^\circ}$

Sol. (D)

$X = \sin\theta + \sin 3\theta + \ldots + \sin 29\theta$

$2(\sin\theta)X = 1 - \cos 2\theta + \cos 2\theta - \cos 4\theta + \ldots + \cos 28\theta - \cos 30\theta$

$X = \frac{1 - \cos 30\theta}{2\sin\theta} = \frac{1}{4\sin 2^\circ}$

25. Let P(3, 2, 6) be a point in space and Q be a point on the line $\vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \mu(-3\hat{i} + \hat{j} + 5\hat{k})$. Then the value of μ for which the vector $\overrightarrow{PQ}$ is parallel to the plane $x - 4y + 3z = 1$ is

(A) $\frac{1}{4}$ (B) $-\frac{1}{4}$

(C) $\frac{1}{8}$ (D) $-\frac{1}{8}$

Sol. (A)

Any point on the line can be taken as

$Q \equiv \{(1 - 3\mu), (\mu - 1), (5\mu + 2)\}$

$\overrightarrow{PQ} = \{-3\mu - 2, \mu - 3, 5\mu - 4\}$

Now, $1(-3\mu - 2) - 4(\mu - 3) + 3(5\mu - 4) = 0$

$\Rightarrow -3\mu - 2 - 4\mu + 12 + 15\mu - 12 = 0$

$8\mu = 2 \Rightarrow \mu = 1/4$.

26. The number of seven digit integers, with sum of the digits equal to 10 and formed by using the digits 1, 2 and 3 only, is

(A) 55 (B) 66

(C) 77 (D) 88

Sol. (C)

Coefficient of x^{10} in $(x + x^2 + x^3)^7$

coefficient of x^3 in $(1 + x + x^2)^7$
coefficient of x^3 in $(1 - x^3)^7 (1 - x)^{-7}$
$= {}^{7+3-1}C_3 - 7$
$= {}^{9}C_3 - 7$
$= \frac{9 \times 8 \times 7}{6} - 7 = 77.$

Alternate:
The digits are 1, 1, 1, 1, 1, 2, 3
or 1, 1, 1, 1, 2, 2, 2
Hence number of seven digit numbers formed
$= \frac{7!}{5!} + \frac{7!}{4!\,3!} = 77.$

27. Let f be a non negative function defined on the interval [0, 1]. If $\int_0^x \sqrt{1-(f'(t))^2}\,dt = \int_0^x f(t)\,dt$, $0 \le x \le 1$, and $f(0) = 0$, then

(A) $f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) > \frac{1}{3}$
(B) $f\left(\frac{1}{2}\right) > \frac{1}{2}$ and $f\left(\frac{1}{3}\right) > \frac{1}{3}$
(C) $f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$
(D) $f\left(\frac{1}{2}\right) > \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$

Sol. (C)
$f' = \pm\sqrt{1-f^2}$
$\Rightarrow f(x) = \sin x$ or $f(x) = -\sin x$ (not possible)
$\Rightarrow f(x) = \sin x$
Also, $x > \sin x \ \forall\ x > 0$.

28. Tangents drawn from the point P (1, 8) to the circle $x^2 + y^2 - 6x - 4y - 11 = 0$ touch the circle at the points A and B. The equation of the circumcircle of the triangle PAB is

(A) $x^2 + y^2 + 4x - 6y + 19 = 0$
(B) $x^2 + y^2 - 4x - 10y + 19 = 0$
(C) $x^2 + y^2 - 2x + 6y - 29 = 0$
(D) $x^2 + y^2 - 6x - 4y + 19 = 0$

Sol. (B)
The centre of the circle is C (3, 2).
Since CA and CB are perpendicular to PA and PB, CP is the diameter of the circumcircle of triangle PAB.
Its equation is
$(x - 3)(x - 1) + (y - 2)(y - 8) = 0$
or $x^2 + y^2 - 4x - 10y + 19 = 0$.

SECTION–II
Multiple Correct Choice Type

This section contains 4 multiple choice questions. Each question has 4 choices (A), (B), (C) and (D) for its answer, out of which **ONE OR MORE** is/are correct.

29. In a triangle ABC with fixed base BC, the vertex A moves such that $\cos B + \cos C = 4\sin^2\frac{A}{2}$. If a, b and c denote the lengths of the sides of the triangle opposite to the angles A, B and C, respectively, then

(A) $b + c = 4a$
(B) $b + c = 2a$
(C) locus of point A is an ellipse
(D) locus of point A is a pair of straight lines

Sol. (B, C)

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$2\cos\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right) - 4\sin^2\frac{A}{2}$

$\cos\left(\frac{B-C}{2}\right) - 2\sin(A/2)$

$\Rightarrow \frac{\cos\left(\frac{B-C}{2}\right)}{\sin A/2} = 2$

$\Rightarrow \frac{\sin B + \sin C}{\sin A} = 2$

$\Rightarrow b + c = 2a$ (constant).

30. If $\frac{\sin^4 x}{2} + \frac{\cos^4 x}{3} = \frac{1}{5}$, then

(A) $\tan^2 x = \frac{2}{3}$

(B) $\frac{\sin^8 x}{8} + \frac{\cos^8 x}{27} = \frac{1}{125}$

(C) $\tan^2 x = \frac{1}{3}$

(D) $\frac{\sin^8 x}{8} + \frac{\cos^8 x}{27} = \frac{2}{125}$

Sol. **(A, B)**

$\frac{\sin^4 x}{2} + \frac{\cos^4 x}{3} = \frac{1}{5}$

$3\sin^4 x + 2(1 - \sin^2 x)^2 = \frac{6}{5}$

$\Rightarrow 25\sin^4 x - 20\sin^2 x + 4 = 0$

$\Rightarrow \sin^2 x = \frac{2}{5}$ and $\cos^2 x = \frac{3}{5}$

$\therefore \tan^2 x = \frac{2}{3}$ and $\frac{\sin^8 x}{8} + \frac{\cos^8 x}{27} = \frac{1}{125}$

31. Let $L = \lim_{x\to 0} \frac{a - \sqrt{a^2 - x^2} - \frac{x^2}{4}}{x^4}$, $a > 0$. If L is finite, then

(A) $a = 2$

(B) $a = 1$

(C) $L = \frac{1}{64}$

(D) $L = \frac{1}{32}$

Sol. **(A, C)**

$L = \lim_{x\to 0} \frac{a - \sqrt{a^2 - x^2} - \frac{x^2}{4}}{x^4} = \lim_{x\to 0} \frac{1}{x^2\left(a + \sqrt{a^2 - x^2}\right)} - \frac{1}{4x^2}$

$= \lim_{x\to 0} \frac{(4-a) - \sqrt{a^2 - x^2}}{4x^2\left(a + \sqrt{a^2 - x^2}\right)}$

numerator $\to 0$ if $a = 2$ and then $L = \frac{1}{64}$.

32. Area of the region bounded by the curve $y = e^x$ and lines $x = 0$ and $y = e$ is

(A) $e - 1$

(B) $\int_1^e \ln(e+1-y)\,dy$

(C) $e - \int_0^1 e^x dx$

(D) $\int_1^e \ln y\,dy$

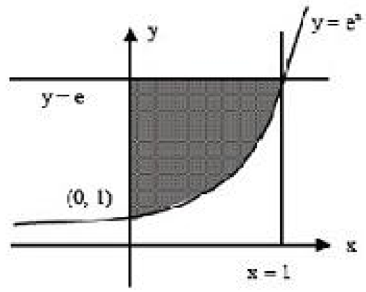
Sol. (B, C, D)

Required Area = $\int_1^e \ln y\,dy$

$= (y\ln y - y)_1^e = (e - e) - \{-1\} = 1.$

Also, $\int_1^e \ln y\,dy = \int_1^e \ln(e+1-y)\,dy$

Further the required area $= e \times 1 - \int_0^1 e^x dx$.



SECTION–III
Comprehension Type

This section contains 2 groups of questions. Each group has 3 multiple choice questions based on a paragraph. Each question has 4 choices (A), (B), (C) and (D) for its answer, out of which **ONLY ONE** is correct.

Paragraph for question Nos. 33 to 35

A fair die is tossed repeatedly until a six is obtained. Let X denote the number of tosses required.

33. The probability that $X = 3$ equals

(A) $\frac{25}{216}$

(B) $\frac{25}{36}$

(C) $\frac{5}{36}$

(D) $\frac{125}{216}$

Sol. (A)

$P(X = 3) = \left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\frac{1}{6} = \frac{25}{216}$.

34. The probability that $X \geq 3$ equals

(A) $\frac{125}{216}$

(B) $\frac{25}{36}$

(C) $\frac{5}{36}$

(D) $\frac{25}{216}$

Sol. (B)

$P(X < 2) = \frac{1}{6} + \frac{5}{6} \times \frac{1}{6} = \frac{11}{36}$

Required probability $= 1 - \frac{11}{36} = \frac{25}{36}$.

35. The conditional probability that $X \geq 6$ given $X > 3$ equals

(A) $\frac{125}{216}$ (B) $\frac{25}{216}$

(C) $\frac{5}{36}$ (D) $\frac{25}{36}$

Sol. **(D)**

For $X \geq 6$, the probability is

$$\frac{5^5}{6^6}+\frac{5^6}{6^7}+...\infty=\frac{5^5}{6^6}\left(\frac{1}{1-5/6}\right)=\left(\frac{5}{6}\right)^5$$

For $X > 3$

$$\frac{5^3}{6^4}+\frac{5^4}{6^5}+\frac{5^5}{6^6}+...\infty=\left(\frac{5}{6}\right)^3$$

Hence the conditional probability $\frac{(5/6)^6}{(5/6)^3}=\frac{25}{36}$.

Paragraph for question Nos. 36 to 38

Let $\mathcal{A}$ be the set of all 3×3 symmetric matrices all of whose entries are either 0 or 1. Five of these entries are 1 and four of them are 0.

36. The number of matrices in $\mathcal{A}$ is

(A) 12 (B) 6

(C) 9 (D) 3

Sol. **(A)**

If two zero's are the entries in the diagonal, then

$^3C_2 \times {}^3C_1$

If all the entries in the principle diagonal is 1, then

$^3C_1$

$\Rightarrow$ Total matrix = 12.

37. The number of matrices A in $\mathcal{A}$ for which the system of linear equations $A\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix}$ has a unique solution, is

(A) less than 4 (B) at least 4 but less than 7

(A) atleast 7 but less than 10 (D) at least 10

Sol. **(B)**

$$\begin{bmatrix}0 & a & b\\ a & 0 & c\\ b & c & 1\end{bmatrix}$$

either $b = 0$ or $c = 0 \Rightarrow |A| \neq 0$

$\Rightarrow$ 2 matrices

$$\begin{bmatrix}0 & a & b\\ a & 1 & c\\ b & c & 0\end{bmatrix}$$

either $a = 0$ or $c = 0 \Rightarrow |A| \neq 0$

$\Rightarrow$ 2 matrices

$$\begin{bmatrix} 1 & a & b \\ a & 0 & c \\ b & c & 0 \end{bmatrix}$$

either $a = 0$ or $b = 0 \rightarrow |A| \neq 0$

$\rightarrow$ 2 matrices.

$$\begin{bmatrix} 1 & a & b \\ a & 1 & c \\ b & c & 1 \end{bmatrix}$$

If $a = b = 0 \rightarrow |A| = 0$

If $a = c = 0 \rightarrow |A| = 0$

If $b = c = 0 \Rightarrow |A| = 0$

$\Rightarrow$ there will be only 6 matrices.

33. The number of matrices A in $\mathscr{A}$ for which the system of linear equations $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is inconsistent, is

(A) 0
(B) more than 2
(C) 2
(D) 1

Sol. **(B)**

The six matrix A for which $|A| = 0$ are

$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \rightarrow$ inconsistent.

$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow$ inconsistent.

$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \Rightarrow$ infinite solutions.

$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow$ inconsistent.

$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \Rightarrow$ inconsistent.

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow$ infinite solutions.

SECTION – IV
Matrix – Match Type

This section contains 2 questions. Each question contains statements given in two columns, which have to be matched. The statement in **Column I** are labelled A, B, C and D, while the statements in **Column II** are labelled p, q, r, s and t. Any given statement in **Column I** can have correct matching with **ONE OR MORE** statement (s) in

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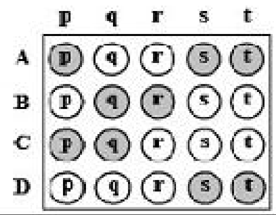
Column II. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example:
If the correct matches are A – p, s and t; B – q and r; C – p and q; and D – s and t; then the correct darkening of bubbles will look like the following.



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39. Match the conics in **Column I** with the statements/expressions in **Column II**.

	Column I		**Column II**
(A)	Circle	(p)	The locus of the point (h, k) for which the line hx + ky = 1 touches the circle $x^2 + y^2 = 4$
(B)	parabola	(q)	Points z in the complex plane satisfying $\lvert z + 2\rvert - \lvert z - 2\rvert = \pm 3$
(C)	Ellipse	(r)	Points of the conic have parametric representation $x = \sqrt{3}\left(\frac{1-t^2}{1+t^2}\right)$, $y = \frac{2t}{1+t^2}$
(D)	Hyperbola	(s)	The eccentricity of the conic lies in the interval $1 \le x < \infty$
		(t)	Points z in the complex plane satisfying $\text{Re}(z + 1)^2 = \lvert z\rvert^2 + 1$

Sol. **(A) → (p)** **(B) → (s, t)** **(C) → (r)** **(D) → (q, s)**

(p). $\frac{1}{k^2} = 4\left(1 + \frac{h^2}{k^2}\right)$

$\Rightarrow 1 = 4(k^2 + h^2)$

$\therefore h^2 + k^2 = \left(\frac{1}{2}\right)^2$ which is a circle.

(q). If $\lvert z - z_1\rvert - \lvert z - z_2\rvert = k$ where $k < \lvert z_1 - z_2\rvert$
the locus is a hyperbola.

(r). Let $t = \tan\alpha$

$\Rightarrow x = \sqrt{3}\cos 2\alpha$ and $y = \sin 2\alpha$

or $\cos 2\alpha = \frac{x}{\sqrt{3}}$ and $\sin 2\alpha = y$

$\therefore \frac{x^2}{3} + y^2 = \sin^2 2\alpha + \cos^2 2\alpha = 1$ which is an ellipse.

(s). If eccentricity is $[1, \infty)$, then the conic can be a parabola (if e = 1) and a hyperbola if $e \in (1, \infty)$.

(t). Let $z = x + iy$; $x, y \in R$

$\Rightarrow (x + 1)^2 - y^2 = x^2 + y^2 + 1$

$\Rightarrow y^2 = x$; which is a parabola.

40. Match the statements/expressions in **Column I** with the open intervals in **Column II**.

	Column I		**Column II**
(A)	Interval contained in the domain of definition of non-zero solutions of the differential equation $(x - 3)^2 y' + y = 0$	(p)	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(B)	Interval containing the value of the integral $\int_1^5 (x-1)(x-2)(x-3)(x-4)(x-5)dx$	(q)	$\left(0, \frac{\pi}{2}\right)$
(C)	Interval in which at least one of the points of local maximum of $\cos^2 x + \sin x$ lies	(r)	$\left(\frac{\pi}{8}, \frac{5\pi}{4}\right)$
(D)	Interval in which $\tan^{-1}(\sin x + \cos x)$ is increasing	(s)	$\left(0, \frac{\pi}{8}\right)$
		(t)	$(-\pi, \pi)$

Sol. **(A) → (p, q, s)** **(B) → (p, t)** **(C) → (p, q, r, t)** **(D) → (s)**

(A). $(x-3)^2 \frac{dy}{dx} + y = 0$

$$\int \frac{dx}{(x-3)^2} = -\int \frac{dy}{y}$$

$$\Rightarrow \frac{1}{x-3} = \ln|y| + c$$

so domain is $R - \{3\}$.

(B). Put $x = t + 3$

$$\int_{-2}^{2} (t+2)(t+1)t(t-1)(t-2)dt = \int_{-2}^{2} t(t^2-1)(t^2-4)dt = 0$$ (being odd function)

(C). $f(x) = \frac{5}{4} - \left(\sin x - \frac{1}{2}\right)^2$

Maximum value occurs when $\sin x = \frac{1}{2}$

(D). $f'(x) > 0$ if $\cos x > \sin x$.