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IIT-JEE

Previous Year

Paper - I

“2010”

(Mathematics)

PART - II: MATHEMATICS

SECTION – I (Single Correct Choice Type)

This Section contains **8 multiple choice questions.** Each question has four choices A), B), C) and D) out of which **ONLY ONE** is correct.

29. Let ω be a complex cube root of unity with $\omega \neq 1$. A fair die is thrown three times. If r_1, r_2 and r_3 are the numbers obtained on the die, then the probability that $\omega^{r_1} + \omega^{r_2} + \omega^{r_3} = 0$ is

(A) $\frac{1}{18}$ (B) $\frac{1}{9}$

(C) $\frac{2}{9}$ (D) $\frac{1}{36}$

Sol. **(C)**

$r_1, r_2, r_3 \in \{1, 2, 3, 4, 5, 6\}$

r_1, r_2, r_3 are of the form $3k$, $3k + 1$, $3k + 2$

Required probability $= \frac{3! \times {}^2C_1 \times {}^2C_1 \times {}^2C_1}{6 \times 6 \times 6} = \frac{6 \times 8}{216} = \frac{2}{9}$.

30. Let P, Q, R and S be the points on the plane with position vectors $-2\hat{i} - \hat{j}$, $4\hat{i}$, $3\hat{i} + 3\hat{j}$ and $-3\hat{i} + 2\hat{j}$ respectively. The quadrilateral PQRS must be a

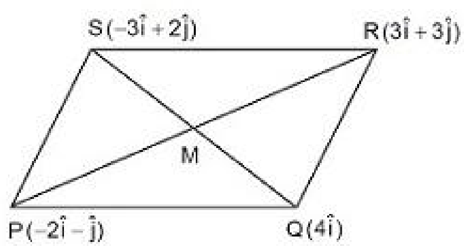
(A) parallelogram, which is neither a rhombus nor a rectangle

(B) square

(C) rectangle, but not a square

(D) rhombus, but not a square

Sol. **(A)**



Evaluating midpoint of PR and QS which gives $M = \left[\frac{\hat{i}}{2} + \hat{j}\right]$, same for both.

$\overline{PQ} = \overline{SR} = 6\hat{i} + \hat{j}$

$\overline{PS} = \overline{QR} = -\hat{i} + 3\hat{j}$

$\Rightarrow \overline{PQ} \cdot \overline{PS} \neq 0$

$\overline{PQ} \parallel \overline{SR}$, $\overline{PS} \parallel \overline{QR}$ and $|\overline{PQ}| = |\overline{SR}|$, $|\overline{PS}| = |\overline{QR}|$

Hence, PQRS is a parallelogram but not rhombus or rectangle.

31. The number of 3 × 3 matrices A whose entries are either 0 or 1 and for which the system $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has exactly two distinct solutions, is

(A) 0 (B) $2^9 - 1$

(C) 168 (D) 2

Sol. **(A)**

Three planes cannot intersect at two distinct points.

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32. The value of $\lim\limits_{x\to 0}\frac{1}{x^3}\int_0^x \frac{t\ln(1+t)}{t^4+4}dt$ is

(A) 0

(B) $\frac{1}{12}$

(C) $\frac{1}{24}$

(D) $\frac{1}{64}$

Sol. **(B)**

$\lim\limits_{x\to 0}\frac{1}{x^3}\int_0^x \frac{t\ln(1+t)dt}{t^4+4} = \lim\limits_{x\to 0}\frac{x\ln(1+x)}{(x^4+4)3x^2}$

$= \lim\limits_{x\to 0}\frac{1}{3}\frac{\ln(1+x)}{x(x^4+4)} = \frac{1}{12}$.

33. Let p and q be real numbers such that $p \neq 0$, $p^3 \neq q$ and $p^3 \neq -q$. If α and β are nonzero complex numbers satisfying $\alpha + \beta = -p$ and $\alpha^3 + \beta^3 = q$, then a quadratic equation having $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ as its roots is

(A) $(p^3+q)x^2-(p^3+2q)x+(p^3+q)=0$

(B) $(p^3+q)x^2-(p^3-2q)x+(p^3+q)=0$

(C) $(p^3-q)x^2-(5p^3-2q)x+(p^3-q)=0$

(D) $(p^3-q)x^2-(5p^3+2q)x+(p^3-q)=0$

Sol. **(B)**

$\alpha^3 + \beta^3 = q$

$\Rightarrow (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta) = q$

$\Rightarrow -p^3 + 3p\alpha\beta = q \Rightarrow \alpha\beta = \frac{q+p^3}{3p}$

$x^2 - \left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)x + \frac{\alpha}{\beta}\cdot\frac{\beta}{\alpha} = 0$

$x^2 - \frac{(\alpha^2+\beta^2)}{\alpha\beta}x + 1 = 0$

$\Rightarrow x^2 - \left(\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}\right)x+1=0$

$\Rightarrow x^2 - \frac{p^2-2\left(\frac{p^3+q}{3p}\right)}{\frac{p^3+q}{3p}}x+1=0$

$\Rightarrow (p^3+q)x^2 - (3p^3-2p^3-2q)x + (p^3+q) = 0$

$\Rightarrow (p^3+q)x^2 - (p^3-2q)x + (p^3+q) = 0$.

34. Let f, g and h be real-valued functions defined on the interval [0, 1] by $f(x) = e^{x^2} + e^{-x^2}$, $g(x) = xe^{x^2} + e^{-x^2}$ and $h(x) = x^2e^{x^2} + e^{-x^2}$. If a, b and c denote, respectively, the absolute maximum of f, g and h on [0, 1], then

(A) $a = b$ and $c \neq b$

(B) $a = c$ and $a \neq b$

(C) $a \neq b$ and $c \neq b$

(D) $a = b = c$

Sol. **(D)**

$f(x) = e^{x^2} + e^{-x^2} \Rightarrow f'(x) = 2x\left(e^{x^2} - e^{-x^2}\right) \geq 0 \ \forall x \in [0, 1]$

Clearly for $0 \leq x \leq 1$ $\quad f(x) \geq g(x) \geq h(x)$

$\because f(1) = g(1) = h(1) = e + \frac{1}{e}$ and f(1) is the greatest

$\therefore a = b = c = e + \frac{1}{e} \Rightarrow a = b = c.$

35. If the angles A, B and C of a triangle are in an arithmetic progression and if a, b and c denote the lengths of the sides opposite to A, B and C respectively, then the value of the expression $\frac{a}{c}\sin 2C + \frac{c}{a}\sin 2A$ is

(A) $\frac{1}{2}$

(B) $\frac{\sqrt{3}}{2}$

(C) 1

(D) $\sqrt{3}$

Sol. **(D)**

$B = 60°$

$\therefore \frac{a}{c}\sin 2C + \frac{c}{a}\sin 2A = 2\sin A\cos C + 2\sin C\cos A$

$= 2\sin(A + C) = 2\sin B = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3}$.

36. Equation of the plane containing the straight line $\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$ and perpendicular to the plane containing the straight lines $\frac{x}{3} = \frac{y}{4} = \frac{z}{2}$ and $\frac{x}{4} = \frac{y}{2} = \frac{z}{3}$ is

(A) $x + 2y - 2z = 0$

(B) $3x + 2y - 2z = 0$

(C) $x - 2y + z = 0$

(D) $5x + 2y - 4z = 0$

Sol. **(C)**

Plane 1: $ax + by + cz = 0$ contains line $\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$

$\therefore 2a + 3b + 4c = 0 \quad$ ...(i)

Plane 2: $a'x + b'y + c'z = 0$ is perpendicular to plane containing lines $\frac{x}{3} = \frac{y}{4} = \frac{z}{2}$ and $\frac{x}{4} = \frac{y}{2} = \frac{z}{3}$

$\therefore 3a' + 4b' + 2c' = 0$ and $4a' + 2b' + 3c' = 0$

$\Rightarrow \quad \frac{a'}{12-4} = \frac{b'}{8-9} = \frac{c'}{6-16}$

$\Rightarrow \quad 8a - b - 10c = 0 \quad$ ...(ii)

From (i) and (ii)

$\frac{a}{-30+4} = \frac{b}{32+20} = \frac{c}{-2-24}$

$\Rightarrow$ Equation of plane $x - 2y + z = 0$.

SECTION – II (Multiple Correct Choice Type)

This section contains **5 multiple choice questions.** Each question has four choices A), B), C) and D) out of which **ONE OR MORE** may be correct.

37. Let z_1 and z_2 be two distinct complex numbers and let $z = (1 - t)\, z_1 + tz_2$ for some real number t with $0 < t < 1$. If Arg (w) denotes the principal argument of a non-zero complex number w, then

(A) $|z - z_1| + |z - z_2| = |z_1 - z_2|$

(B) $\text{Arg}\,(z - z_1) = \text{Arg}\,(z - z_2)$

(C) $\begin{vmatrix} z - z_1 & \bar{z} - \bar{z}_1 \\ z_2 - z_1 & \bar{z}_2 - \bar{z}_1 \end{vmatrix} = 0$

(D) $\text{Arg}\,(z - z_1) = \text{Arg}\,(z_2 - z_1)$

Sol. **(A), (C), (D)**

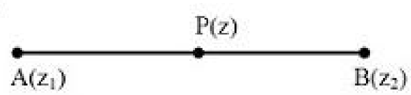
Given $z = (1 - t)\, z_1 + tz_2$

$\Rightarrow \dfrac{z - z_1}{z_2 - z_1} = t \Rightarrow \arg\left(\dfrac{z - z_1}{z_2 - z_1}\right) = 0$ … (1)

$\Rightarrow \arg\,(z - z_1) = \arg\,(z_2 - z_1)$

$\dfrac{z - z_1}{z_2 - z_1} = \dfrac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1}$

$\begin{vmatrix} z - z_1 & \bar{z} - \bar{z}_1 \\ z_2 - z_1 & \bar{z}_2 - \bar{z}_1 \end{vmatrix} = 0$



AP + PB = AB

$\Rightarrow |z - z_1| + |z - z_2| = |z_1 - z_2|$.

38. The value(s) of $\displaystyle\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx$ is (are)

(A) $\dfrac{22}{7} - \pi$

(B) $\dfrac{2}{105}$

(C) 0

(D) $\dfrac{71}{15} - \dfrac{3\pi}{2}$

Sol. **(A)**

$\displaystyle\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx$

$= \displaystyle\int_0^1 \left(x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2}\right)dx$

$= \left[\dfrac{x^7}{7} - \dfrac{2x^6}{3} + x^5 - \dfrac{4x^3}{3} + 4x\right]_0^1 - \pi$

$= \dfrac{1}{7} - \dfrac{2}{3} + 1 - \dfrac{4}{3} + 4 - \pi = \dfrac{22}{7} - \pi$

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39. Let ABC be a triangle such that $\angle ACB = \frac{\pi}{6}$ and let a, b and c denote the lengths of the sides opposite to A, B and C respectively. The value(s) of x for which $a = x^2 + x + 1$, $b = x^2 - 1$ and $c = 2x + 1$ is (are)

(A) $-(2+\sqrt{3})$
(B) $1 + \sqrt{3}$
(C) $2 + \sqrt{3}$
(D) $4\sqrt{3}$

Sol. **(B)**

Using cosine rule for $\angle C$

$$\frac{\sqrt{3}}{2} = \frac{(x^2+x+1)^2 + (x^2-1)^2 - (2x+1)^2}{2(x^2+x+1)(x^2-1)}$$

$$\Rightarrow \sqrt{3} = \frac{2x^2+2x-1}{x^2+x+1}$$

$$\Rightarrow (\sqrt{3}-2)x^2 + (\sqrt{3}-2)x + (\sqrt{3}+1) = 0$$

$$\Rightarrow x = \frac{(2-\sqrt{3}) \pm \sqrt{3}}{2(\sqrt{3}-2)}$$

$\Rightarrow x = -(2+\sqrt{3}),\ 1 + \sqrt{3} \Rightarrow x = 1 + \sqrt{3}$ as $(x > 0)$.

40. Let A and B be two distinct points on the parabola $y^2 = 4x$. If the axis of the parabola touches a circle of radius r having AB as its diameter, then the slope of the line joining A and B can be

(A) $-\frac{1}{r}$
(B) $\frac{1}{r}$
(C) $\frac{2}{r}$
(D) $-\frac{2}{r}$

Sol. **(C), (D)**

$A = (t_1^2,\ 2t_1)$, $B = (t_2^2,\ 2t_2)$

$$\text{Centre} = \left[\frac{t_1^2+t_2^2}{2},\ (t_1+t_2)\right]$$

$t_1 + t_2 = \pm r$

$$m = \frac{2(t_1-t_2)}{t_1^2-t_2^2} = \frac{2}{t_1+t_2} = \pm\frac{2}{r}.$$

41. Let f be a real-valued function defined on the interval $(0, \infty)$ by $f(x) = \ln x + \int_0^x \sqrt{1+\sin t}\, dt$. Then which of the following statement(s) is (are) true?

(A) $f''(x)$ exists for all $x \in (0, \infty)$
(B) $f'(x)$ exists for all $x \in (0, \infty)$ and f' is continuous on $(0, \infty)$, but not differentiable on $(0, \infty)$
(C) there exists $\alpha > 1$ such that $|f'(x)| < |f(x)|$ for all $x \in (\alpha, \infty)$
(D) there exists $\beta > 0$ such that $|f(x)| + |f'(x)| \le \beta$ for all $x \in (0, \infty)$

Sol. **(B), (C)**

$$f'(x) = \frac{1}{x} + \sqrt{1+\sin x}$$

$f'(x)$ is not differentiable at $\sin x = -1$ or $x = 2n\pi - \frac{\pi}{2}, n \in N$

In $x \in (1, \infty)$ $f(x) > 0$, $f'(x) > 0$

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Consider f(x) – f′(x)

$= \ln x + \int_0^x \sqrt{1+\sin t}\,dt - \frac{1}{x} - \sqrt{1+\sin x}$

$= \left(\int_0^x \sqrt{1+\sin t}\,dt - \sqrt{1+\sin x}\right) + \ln x - \frac{1}{x}$

Consider $g(x) = \int_0^x \sqrt{1+\sin t}\,dt - \sqrt{1+\sin x}$

It can be proved that $g(x) \ge 2\sqrt{2} - \sqrt{10} \quad \forall\, x \in (0, \infty)$

Now there exists some $\alpha > 1$ such that $\frac{1}{x} - \ln x \le 2\sqrt{2} - \sqrt{10}$ for all $x \in (\alpha, \infty)$ as $\frac{1}{x} - \ln x$ is strictly decreasing function.

$\Rightarrow g(x) \ge \frac{1}{x} - \ln x$.

SECTION – III (Paragraph Type)

This section contains **2 paragraphs.** Based upon the first paragraph 2 multiple choice questions and based upon the second paragraph **3 multiple choice questions** have to be answered. Each of these questions has four choices A), B), C) and D) out of **WHICH ONLY ONE CORRECT.**

Paragraph for Questions 42 to 43

The circle $x^2 + y^2 - 8x = 0$ and hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ intersect at the points A and B.

42. Equation of a common tangent with positive slope to the circle as well as to the hyperbola is

(A) $2x - \sqrt{5}y - 20 = 0$ (B) $2x - \sqrt{5}y + 4 = 0$

(C) $3x - 4y + 8 = 0$ (D) $4x - 3y + 4 = 0$

Sol. **(B)**

A tangent to $\frac{x^2}{9} - \frac{y^2}{4} = 1$ is $y = mx + \sqrt{9m^2 - 4}$, $m > 0$

It is tangent to $x^2 + y^2 - 8x = 0$

$\therefore \frac{4m + \sqrt{9m^2 - 4}}{\sqrt{1+m^2}} = 4$

$\Rightarrow 495m^4 + 104m^2 - 400 = 0$

$\Rightarrow m^2 = \frac{4}{5}$ or $m = \frac{2}{\sqrt{5}}$

$\therefore$ the tangent is $y = \frac{2}{\sqrt{5}}m + \frac{4}{\sqrt{5}}$

$\Rightarrow 2x - \sqrt{5}y + 4 = 0$.

43. Equation of the circle with AB as its diameter is

(A) $x^2 + y^2 - 12x + 24 = 0$ (B) $x^2 + y^2 + 12x + 24 = 0$

(C) $x^2 + y^2 + 24x - 12 = 0$ (D) $x^2 + y^2 - 24x - 12 = 0$

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Consider f(x) – f′(x)

$= \ln x + \int_0^x \sqrt{1+\sin t}\, dt - \frac{1}{x} - \sqrt{1+\sin x}$

$= \left(\int_0^x \sqrt{1+\sin t}\, dt - \sqrt{1+\sin x} \right) + \ln x - \frac{1}{x}$

Consider $g(x) = \int_0^x \sqrt{1+\sin t}\, dt - \sqrt{1+\sin x}$

It can be proved that $g(x) \ge 2\sqrt{2} - \sqrt{10}\ \ \forall x \in (0, \infty)$

Now there exists some $\alpha > 1$ such that $\frac{1}{x} - \ln x \le 2\sqrt{2} - \sqrt{10}$ for all $x \in (\alpha, \infty)$ as $\frac{1}{x} - \ln x$ is strictly decreasing function.

$\Rightarrow g(x) \ge \frac{1}{x} - \ln x$.

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(C) $3x - 4y + 8 = 0$ (D) $4x - 3y + 4 = 0$

Sol. **(B)**

A tangent to $\frac{x^2}{9} - \frac{y^2}{4} = 1$ is $y = mx + \sqrt{9m^2 - 4}, m > 0$

It is tangent to $x^2 + y^2 - 8x = 0$

$\therefore \frac{4m + \sqrt{9m^2 - 4}}{\sqrt{1+m^2}} = 4$

$\Rightarrow 495m^4 + 104m^2 - 400 = 0$

$\Rightarrow m^2 = \frac{4}{5}$ or $m = \frac{2}{\sqrt{5}}$

$\therefore$ the tangent is $y = \frac{2}{\sqrt{5}}m + \frac{4}{\sqrt{5}}$

$\Rightarrow 2x - \sqrt{5}y + 4 = 0$.

43. Equation of the circle with AB as its diameter is

(A) $x^2 + y^2 - 12x + 24 = 0$ (B) $x^2 + y^2 + 12x + 24 = 0$

(C) $x^2 + y^2 + 24x - 12 = 0$ (D) $x^2 + y^2 - 24x - 12 = 0$

Sol. **(A)**

A point on hyperbola is $(3\sec\theta, 2\tan\theta)$

It lies on the circle, so $9\sec^2\theta + 4\tan^2\theta - 24\sec\theta = 0$

$\Rightarrow 13\sec^2\theta - 24\sec\theta - 4 = 0 \Rightarrow \sec\theta = 2, -\frac{2}{13}$

$\therefore \sec\theta = 2 \Rightarrow \tan\theta = \sqrt{3}$.

The point of intersection are $A(6, 2\sqrt{3})$ and $B(6, -2\sqrt{3})$

$\therefore$ The circle with AB as diameter is

$(x-6)^2 + y^2 = (2\sqrt{3})^2 \Rightarrow x^2 + y^2 - 12x + 24 = 0.$

Paragraph for Questions 44 to 46

Let p be an odd prime number and T_p be the following set of 2×2 matrices :

$$T_p = \left\{A = \begin{bmatrix} a & b \\ c & a \end{bmatrix} : a, b, c \in \{0, 1, \ldots, p-1\}\right\}$$

44. The number of A in T_p such that A is either symmetric or skew-symmetric or both, and det(A) divisible by p is

(A) $(p-1)^2$
(B) $2(p-1)$
(C) $(p-1)^2 + 1$
(D) $2p - 1$

Sol. **(D)**

We must have $a^2 - b^2 = kp$

$\Rightarrow (a+b)(a-b) = kp$

$\Rightarrow$ either $a - b = 0$ or $a + b$ is a multiple of p

when $a = b$; number of matrices is p

and when $a + b$ = multiple of p $\Rightarrow$ a, b has $p - 1$

$\therefore$ Total number of matrices $= p + p - 1$

$= 2p - 1$.

45. The number of A in T_p such that the trace of A is not divisible by p but det (A) is divisible by p is
[Note: The trace of a matrix is the sum of its diagonal entries.]

(A) $(p-1)(p^2 - p + 1)$
(B) $p^3 - (p-1)^2$
(C) $(p-1)^2$
(D) $(p-1)(p^2 - 2)$

Ans. **(C)**

46. The number of A in T_p such that det (A) is not divisible by p is

(A) $2p^2$
(B) $p^3 - 5p$
(C) $p^3 - 3p$
(D) $p^3 - p^2$

Ans. **(D)**

SECTION – IV (Integer Type)

This section contains **TEN** questions. The answer to each question is a single digit integer ranging from 0 to 9. The correct digit below the question number in the **ORS** is to be bubbled.

47. Let S_k, k = 1, 2, ….. , 100, denote the sum of the infinite geometric series whose first term is $\frac{k-1}{k!}$ and the common ratio is $\frac{1}{k}$. Then the value of $\frac{100^2}{100!}+\sum_{k=1}^{100}\left|(k^2-3k+1)S_k\right|$ is

Sol. **(3)**

$$S_k = \frac{\frac{k-1}{k!}}{1-\frac{1}{k}} = \frac{1}{(k-1)!}$$

$$\sum_{k=2}^{100}\left|(k^2-3k+1)\frac{1}{(k-1)!}\right|$$

$$= \sum_{k=2}^{100}\left|\frac{(k-1)^2-k}{(k-1)!}\right|$$

$$= \sum\left|\frac{k-1}{(k-2)!}-\frac{k}{(k-1)!}\right|$$

$$= \left|\frac{2}{1!}-\frac{3}{2!}\right|+\left|\frac{3}{2!}-\frac{4}{3!}\right|+\cdots$$

$$= \frac{2}{1!}-\frac{1}{0!}+\frac{2}{1!}-\frac{3}{2!}+\frac{3}{2!}-\frac{4}{3!}+\cdots+\frac{99}{98!}-\frac{100}{99!}$$

$$= 3-\frac{100}{99!}.$$

48. The number of all possible values of θ, where 0 < θ < π, for which the system of equations

$$(y + z) \cos 3\theta = (xyz) \sin 3\theta$$

$$x\sin 3\theta = \frac{2\cos 3\theta}{y}+\frac{2\sin 3\theta}{z}$$

$$(xyz) \sin 3\theta = (y + 2z) \cos 3\theta + y \sin 3\theta$$

have a solution (x_0, y_0, z_0) with $y_0z_0 \neq 0$, is

Sol. **(3)**

$(y + z) \cos 3\theta – (xyz) \sin 3\theta = 0$ … (1)

$xyz \sin 3\theta = (2 \cos 3\theta) z + (2 \sin 3\theta) y$ … (2)

$\therefore (y + z) \cos 3\theta = (2 \cos 3\theta) z + (2 \sin 3\theta) y = (y + 2z) \cos 3\theta + y \sin 3\theta$

$y (\cos 3\theta – 2 \sin 3\theta) = z \cos 3\theta$ and

$y (\sin 3\theta – \cos 3\theta) = 0 \Rightarrow \sin 3\theta – \cos 3\theta = 0 \Rightarrow \sin 3\theta = \cos 3\theta$

$\therefore 3\theta = n\pi + \pi/4$

49. Let f be a real-valued differentiable function on R (the set of all real numbers) such that f(1) = 1. If the y-intercept of the tangent at any point P(x, y) on the curve y = f(x) is equal to the cube of the abscissa of P, then the value of f(–3) is equal to

Sol. **(9)**

$y - y_1 = m(x - x_1)$

Put x = 0, to get y intercept

$y_1 - mx_1 = x_1^3$

$y_1 - x_1 \frac{dy}{dx} = x_1^3$

$x\frac{dy}{dx} - y = -x^3$

$\frac{dy}{dx} - \frac{y}{x} = -x^2$

$e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$

$y \times \frac{1}{x} = \int -x^2 \times \frac{1}{x}dx$

$\frac{y}{x} = -\int xdx \Rightarrow \frac{y}{x} = -\frac{x^2}{2} + c$

$\Rightarrow f(x) = -\frac{x^3}{2} + \frac{3}{2}x \;\; \therefore \; f(-3) = 9.$

50. The number of values of θ in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\theta \neq \frac{n\pi}{5}$ for n = 0, ±1, ±2 and $\tan\theta = \cot 5\theta$ as well as $\sin 2\theta = \cos 4\theta$ is

Sol. **(3)**

$\tan\theta = \cot 5\theta$

$\Rightarrow \cos 6\theta = 0$

$4\cos^3 2\theta - 3\cos 2\theta = 0$

$\Rightarrow \cos 2\theta = 0 \text{ or } \pm\frac{\sqrt{3}}{2}$

$\sin 2\theta = \cos 4\theta$

$\Rightarrow 2\sin^2 2\theta + \sin 2\theta - 1 = 0$

$2\sin^2 2\theta + 2\sin 2\theta - \sin 2\theta - 1 = 0$

$\sin 2\theta = -1 \text{ or } \sin 2\theta = \frac{1}{2}$

$\cos 2\theta = 0 \text{ and } \sin 2\theta = -1$

$\Rightarrow 2\theta = -\frac{\pi}{2} \;\; \Rightarrow \theta = -\frac{\pi}{4}$

$\cos 2\theta = \pm\frac{\sqrt{3}}{2}, \; \sin 2\theta = \frac{1}{2}$

$\Rightarrow 2\theta = \frac{\pi}{6}, \frac{5\pi}{6} \;\; \Rightarrow \theta = \frac{\pi}{12}, \frac{5\pi}{12}$

$\therefore \; \theta = -\frac{\pi}{4}, \frac{\pi}{12}, \frac{5\pi}{12}$

51. The maximum value of the expression $\frac{1}{\sin^2\theta + 3\sin\theta\cos\theta + 5\cos^2\theta}$ is

Sol. **(2)**

$\frac{1}{4\cos^2\theta+1+\frac{3}{2}\sin 2\theta}$

$\Rightarrow \frac{1}{2[1+\cos 2\theta]+1+\frac{3}{2}\sin 2\theta}$

lies between $\frac{1}{2}$ to $\frac{11}{2}$

$\therefore$ maximum value is 2.

Minimum value of $1 + 4\cos^2\theta + 3\sin\theta\cos\theta$

$1 + \frac{4(1+\cos 2\theta)}{2}+\frac{3}{2}\sin 2\theta$

$= 1+ 2 + 2\cos 2\theta + \frac{3}{2}\sin 2\theta$

$= 3 + 2\cos 2\theta + \frac{3}{2}\sin 2\theta$

$\therefore = 3 - \sqrt{4+\frac{9}{4}} = 3 - \frac{5}{2} = \frac{1}{2}$.

So maximum value of $\frac{1}{4\cos^2\theta+1+\frac{3}{2}\sin 2\theta}$ is 2.

52. If $\vec{a}$ and $\vec{b}$ are vectors in space given by $\vec{a}=\frac{\hat{i}-2\hat{j}}{\sqrt{5}}$ and $\vec{b}=\frac{2\hat{i}+\hat{j}+3\hat{k}}{\sqrt{14}}$, then the value of $(2\vec{a}+\vec{b})\cdot[(\vec{a}\times\vec{b})\times(\vec{a}-2\vec{b})]$ is

Sol. **(5)**

$E = (2\vec{a}+\vec{b})\cdot\left[2|\vec{b}|^2\vec{a}-2(\vec{a}\cdot\vec{b})\vec{b}-(\vec{a}\cdot\vec{b})\vec{a}+|\vec{a}|^2\vec{b}\right]$

$\vec{a}\cdot\vec{b}=\frac{2-2}{\sqrt{70}}=0$

$|\vec{a}| = 1$

$|b| = 1$

$\vec{a}\cdot\vec{b}=0$

$E = (2\vec{a}+\vec{b})\cdot\left[2|\vec{b}|^2\vec{a}+|\vec{a}|^2\vec{b}\right]$

$= 4|\vec{a}|^2|\vec{b}|^2+|\vec{a}|^2(\vec{a}\cdot\vec{b})+2|\vec{b}|^2(\vec{b}\cdot\vec{a})+|\vec{a}|^2|\vec{b}|^2$

$= 5|\vec{a}|^2|\vec{b}|^2 = 5$

53. The line $2x + y = 1$ is tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. If this line passes through the point of intersection of the nearest directrix and the x-axis, then the eccentricity of the hyperbola is

Sol. **(2)**

Substituting $\left(\frac{a}{e}, 0\right)$ in $y = -2x + 1$

$0 = -\frac{2a}{e}+1$

$\frac{2a}{e} = 1$

$a = \frac{e}{2}$

Also, $1 = \sqrt{a^2m^2 - b^2}$

$1 = a^2m^2 - b^2$

$1 = 4a^2 - b^2$

$1 = \frac{4e^2}{4} - b^2$

$b^2 = e^2 - 1.$

Also, $b^2 = a^2 (e^2 - 1)$

$\therefore$ a = 1, e = 2

54. If the distance between the plane Ax – 2y + z = d and the plane containing the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ is $\sqrt{6}$, then |d| is

Sol. **(6)**

$2l + 3m + 4n = 0$

$3l + 4m + 5n = 0$

$\frac{l}{-1} = \frac{m}{2} = \frac{n}{-1}$

Equation of plane will be

$a (x - 1) + b (y - 2) + c (z - 3) = 0$

$- 1 (x - 1) + 2 (y - 2) - 1 (z - 3) = 0$

$- x + 1 + 2y - 4 - z + 3 = 0$

$- x + 2y - z = 0$

$x - 2y + z = 0$

$\frac{|d|}{\sqrt{6}} = \sqrt{6}$

d = 6.

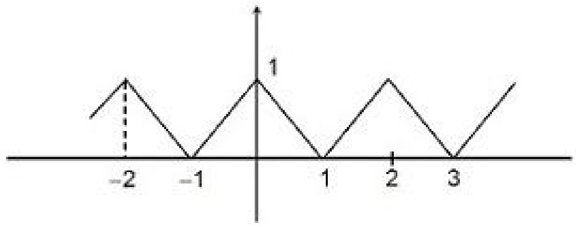
55. For any real number x, let |x| denote the largest integer less than or equal to x. Let f be a real valued function defined on the interval [–10, 10] by

$$f(x) = \begin{cases} x-[x] & \text{if } [x] \text{ is odd} \\ 1+[x]-x & \text{if } [x] \text{ is even} \end{cases}$$

Then the value of $\frac{\pi^2}{10}\int_{-10}^{10} f(x)\ \cos \pi x\ dx$ is

Sol. **(4)**

$$f(x) = \begin{cases} x-1, & 1 \le x < 2 \\ 1-x, & 0 \le x < 1 \end{cases}$$



f(x) is periodic with period 2

$\therefore I = \int_{-10}^{10} f(x)\cos \pi x \, dx$

$= 2\int_{0}^{10} f(x)\cos \pi x \, dx = 2\times 5\int_{0}^{2} f(x)\cos \pi x \, dx$

$= 10\left[\int_{0}^{1}(1-x)\cos \pi x \, dx + \int_{1}^{2}(x-1)\cos \pi x \, dx\right] = 10(I_1 + I_2)$

$I_2 = \int_{1}^{2}(x-1)\cos \pi x \, dx$ put $x - 1 = t$

$I_2 = -\int_{0}^{1} t \cos \pi t \, dt$

$I_1 = \int_{0}^{1}(1-x)\cos \pi x \, dx = -\int_{0}^{1} x\cos(\pi x)dx$

$\therefore I = 10\left[-2\int_{0}^{1} x\cos \pi x \, dx\right]$

$= -20\left[x\frac{\sin \pi x}{\pi} + \frac{\cos \pi x}{\pi^2}\right]_0^1$

$= -20\left[-\frac{1}{\pi^2} - \frac{1}{\pi^2}\right] = \frac{40}{\pi^2} \quad \therefore \frac{\pi^2}{10} I = 4$

56. Let ω be the complex number $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$. Then the number of distinct complex numbers z satisfying

$$\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0 \text{ is equal to}$$

Sol. **(1)**

$\omega = e^{i2\pi/3}$

$$\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$$

$$z\begin{vmatrix} 1 & \omega & \omega^2 \\ 1 & z+\omega^2 & 1 \\ 1 & 1 & z+\omega \end{vmatrix} = 0$$

$\Rightarrow z\left[(z+\omega^2)(z+\omega)-1-\omega(z+\omega-1)+\omega^2(1-z-\omega^2)\right]=0$

$\Rightarrow z^3 = 0$

$\Rightarrow z = 0$ is only solution.

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