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PART II: MATHEMATICS

SECTION–I

Single Correct Choice Type

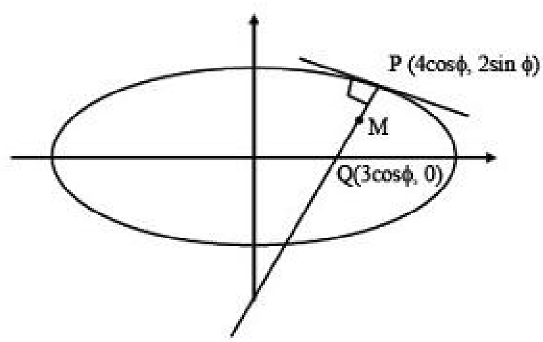
This section contains 4 multiple choice questions. Each question has 4 choices (A), (B), (C) and (D) for its answer, out of which **ONLY ONE** is correct.

20. The normal at a point P on the ellipse $x^2 + 4y^2 = 16$ meets the x-axis at Q. If M is the mid point of the line segment PQ, then the locus of M intersects the latus rectums of the given ellipse at the points

(A) $\left(\pm\frac{3\sqrt{5}}{2}, \pm\frac{2}{7}\right)$ (B) $\left(\pm\frac{3\sqrt{5}}{2}, \pm\frac{\sqrt{19}}{4}\right)$

(C) $\left(\pm 2\sqrt{3}, \pm\frac{1}{7}\right)$ (D) $\left(\pm 2\sqrt{3}, \pm\frac{4\sqrt{3}}{7}\right)$

Sol. **(C)**



Normal is $4x \sec\phi - 2y \operatorname{cosec}\phi = 12$

$Q = (3\cos\phi, 0)$

$M = (\alpha, \beta)$

$\alpha = \frac{3\cos\phi + 4\cos\phi}{2} = \frac{7}{2}\cos\phi$

$\Rightarrow \cos\phi = \frac{2}{7}\alpha$

$\beta = \sin\phi$

$\cos^2\phi + \sin^2\phi = 1$

$\Rightarrow \frac{4}{49}\alpha^2 + \beta^2 = 1 \Rightarrow \frac{4}{49}x^2 + y^2 = 1$

$\Rightarrow$ latus rectum $x = \pm\, 2\sqrt{3}$

$\frac{48}{49} + y^2 = 1 \Rightarrow y = \pm\,\frac{1}{7}$

$(\pm 2\sqrt{3}, \pm 1/7)$.

21. The locus of the orthocentre of the triangle formed by the lines $(1 + p)x - py + p(1 + p) = 0$, $(1 + q)x - qy + q(1 + q) = 0$ and $y = 0$, where $p \neq q$, is

(A) a hyperbola (B) a parabola

(C) an ellipse (D) a straight line

Sol. **(D)**

Intersection point of $y = 0$ with first line is $B(-p, 0)$

Intersection point of $y = 0$ with second line is $A(-q, 0)$

Intersection point of the two lines is $C(pq, (p + 1)(q + 1))$

Altitude from C to AB is $x = pq$

Altitude from B to AC is $y = -\frac{q}{1+q}(x + p)$

Solving these two we get $x = pq$ and $y = -pq$

$\therefore$ locus of orthocentre is $x + y = 0$.

22. A line with positive direction cosines passes through the point P(2, – 1, 2) and makes equal angles with the coordinate axes. The line meets the plane 2x + y + z = 9 at point Q. The length of the line segment PQ equals

(A) 1 (B) $\sqrt{2}$

(C) $\sqrt{3}$ (D) 2

Sol. **(C)**

D.C of the line are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

Any point on the line at a distance t from P(2, – 1, 2) is $\left(2+\frac{t}{\sqrt{3}}, -1+\frac{t}{\sqrt{3}}, 2+\frac{t}{\sqrt{3}}\right)$

which lies on $2x + y + z = 9 \Rightarrow t = \sqrt{3}$.

23. If the sum of first n terms of an A.P. is cn^2, then the sum of squares of these n terms is

(A) $\frac{n(4n^2-1)c^2}{6}$ (B) $\frac{n(4n^2+1)c^2}{3}$

(C) $\frac{n(4n^2-1)c^2}{3}$ (D) $\frac{n(4n^2+1)c^2}{6}$

Sol. **(C)**

$t_n = c\ \{n^2 - (n-1)^2\}$

$= c\ (2n - 1)$

$\Rightarrow t_n^2 = c^2(4n^2-4n+1)$

$\Rightarrow \sum_{n=1}^{n} t_n^2 = c^2\left\{\frac{4n(n+1)(2n+1)}{6} - \frac{4n(n+1)}{2} + n\right\}$

$= \frac{c^2 n}{6}\{4(n+1)(2n+1) - 12(n+1) + 6\}$

$= \frac{c^2 n}{3}\{4n^2 + 6n + 2 - 6n - 6 + 3\} = \frac{c^2}{3}n(4n^2-1)$.

SECTION–II

Multiple Correct Choice Type

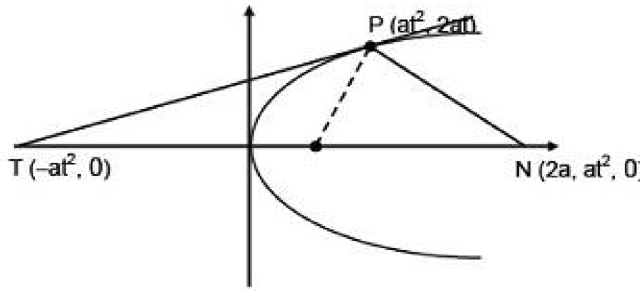
This section contains 5 multiple choice questions. Each question has 4 choices (A), (B), (C) and (D) for its answer, out of which **ONE OR MORE** is/are correct.

24. The tangent PT and the normal PN to the parabola $y^2 = 4ax$ at a point P on it meet its axis at points T and N, respectively. The locus of the centroid of the triangle PTN is a parabola whose

(A) vertex is $\left(\frac{2a}{3}, 0\right)$ (B) directrix is x = 0

(C) latus rectum is $\frac{2a}{3}$ (D) focus is (a, 0)

Sol. **(A, D)**



$G = (h, k)$

$\Rightarrow h = \frac{2a + at^2}{3}, \; k = \frac{2at}{3}$

$\Rightarrow \left(\frac{3h - 2a}{a}\right) = \frac{9k^2}{4a^2}$

$\Rightarrow$ required parabola is

$\frac{9y^2}{4a^2} = \frac{(3x - 2a)}{a} = \frac{3}{a}\left(x - \frac{2a}{3}\right)$

$\Rightarrow y^2 = \frac{4a}{3}\left(x - \frac{2a}{3}\right)$

Vertex $= \left(\frac{2a}{3}, \; 0\right)$; Focus $= (a, 0)$

25. For function $f(x) = x\cos\frac{1}{x}, \; x \geq 1,$

(A) for atleast one x in interval $[1, \infty)$, $f(x + 2) - f(x) < 2$

(B) $\lim\limits_{x \to \infty} f'(x) = 1$

(C) for all x in the interval $[1, \infty)$, $f(x + 2) - f(x) > 2$

(D) $f'(x)$ is strictly decreasing in the interval $[1, \infty)$

Sol. **(B, C, D)**

For $f(x) = x\cos\left(\frac{1}{x}\right), \; x \geq 1$

$f'(x) = \cos\left(\frac{1}{x}\right) + \frac{1}{x}\sin\left(\frac{1}{x}\right) \to 1$ for $x \to \infty$

also $f''(x) = \frac{1}{x^2}\sin\left(\frac{1}{x}\right) - \frac{1}{x^2}\sin\left(\frac{1}{x}\right) - \frac{1}{x^3}\cos\left(\frac{1}{x}\right)$

$= -\frac{1}{x^3}\cos\left(\frac{1}{x}\right) < 0 \quad$ for $x \geq 1$

$\Rightarrow f'(x)$ is decreasing for $[1, \infty)$

$\Rightarrow f'(x + 2) < f'(x)$. Also, $\lim\limits_{x \to \infty} f(x+2) - f(x) = \lim\limits_{x \to \infty}\left[(x+2)\cos\frac{1}{x+2} - x\cos\frac{1}{x}\right] = 2$

$\therefore f(x+2) - f(x) > 2 \; \forall \; x \geq 1$

26. For $0 < \theta < \frac{\pi}{2}$, the solution(s) of $\sum\limits_{m=1}^{6} \operatorname{cosec}\left(\theta + \frac{(m-1)\pi}{4}\right)\operatorname{cosec}\left(\theta + \frac{m\pi}{4}\right) = 4\sqrt{2}$ is(are)

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{6}$

(C) $\frac{\pi}{12}$ (D) $\frac{5\pi}{12}$

Sol. **(C, D)**

Given solutions

$\frac{1}{\sin(\pi/4)}\left[\frac{\sin(\theta + \pi/4 - \theta)}{\sin\theta \cdot \sin(\theta + \pi/4)} + \frac{\sin(\theta + \pi/2 - (\theta + \pi/4))}{\sin(\theta + \pi/4)\cdot\sin(\theta + \pi/2)} + \ldots + \frac{\sin((\theta + 3\pi/2) - (\theta + 5\pi/4))}{\sin(\theta + 3\pi/2)\cdot\sin(\theta + 5\pi/4)}\right] = 4\sqrt{2}$

$\Rightarrow \sqrt{2}[\cot\theta - \cot(\theta + \pi/4) + \cot(\theta + \pi/4) - \cot(\theta + \pi/2) + \ldots + \cot(\theta + 5\pi/4) - \cot(\theta + 3\pi/2)] = 4\sqrt{2}$

$\Rightarrow \tan\theta + \cot\theta = 4 \Rightarrow \tan\theta = 2 \pm \sqrt{3}$

$\Rightarrow \theta = \frac{\pi}{12}$ or $\frac{5\pi}{12}$.

27. An ellipse intersects the hyperbola $2x^2 - 2y^2 = 1$ orthogonally. The eccentricity of the ellipse is reciprocal of that of the hyperbola. If the axes of the ellipse are along the coordinates axes, then

(A) equation of ellipse is $x^2 + 2y^2 = 2$

(B) the foci of ellipse are $(\pm 1, 0)$

(C) equation of ellipse is $x^2 + 2y^2 = 4$

(D) the foci of ellipse are $(\pm \sqrt{2}, 0)$

Sol. **(A, B)**

Ellipse and hyperbola will be confocal

$\Rightarrow (\pm ae, 0) \equiv (\pm 1, 0)$

$\Rightarrow \left(\pm a \times \frac{1}{\sqrt{2}},\ 0\right) \equiv (\pm 1, 0)$

$\Rightarrow a = \sqrt{2}$ and $e = \frac{1}{\sqrt{2}}$

$\Rightarrow b^2 = a^2(1 - e^2) \Rightarrow b^2 = 1$

$\therefore$ Equation of ellipse $\frac{x^2}{2} + \frac{y^2}{1} = 1$.

28. If $I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{(1+\pi^x)\sin x}dx$, $n = 0, 1, 2, \ldots$, then

(A) $I_n = I_{n+2}$

(B) $\sum_{m=1}^{10} I_{2m+1} = 10\pi$

(C) $\sum_{m=1}^{10} I_{2m} = 0$

(D) $I_n = I_{n+1}$

Sol. **(A, B, C)**

$$I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{(1+\pi^x)\sin x}dx$$

$$= \int_0^{\pi}\left(\frac{\sin nx}{(1+\pi^x)\sin x} + \frac{\pi^x \sin nx}{(1+\pi^x)\sin x}\right)dx = \int_0^{\pi} \frac{\sin nx}{\sin x}$$

$$\text{Now, } I_{n+2} - I_n = \int_0^{\pi} \frac{\sin(n+2)x - \sin nx}{\sin x}dx$$

$$= \int_0^{\pi} \frac{2\cos(n+1)x \cdot \sin x}{\sin x}dx = 0$$

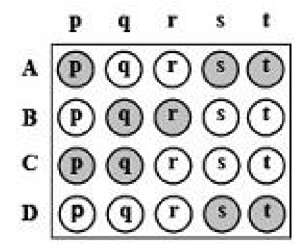
$$\Rightarrow I_1 = \pi,\ I_2 = \int_0^{\pi} 2\cos x\, dx = 0$$

SECTION – III

Matrix – Match Type

This section contains 2 questions. Each question contains statements given in two columns, which have to be matched. The statement in **Column I** are labelled A, B, C and D, while the statements in **Column II** are labelled p, q, r, s and t. Any given statement in **Column I** can have correct matching with **ONE OR MORE** statement (s) in **Column II**. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example:

If the correct matches are A – p, s and t; B – q and r; C – p and q; and D –s and t; then the correct darkening of bubbles will look like the following.



29. Match the statements/expressions in **Column I** with the values given in **Column II.**

	Column I		Column II
(A)	The number of solutions of the equation $xe^{\sin x} - \cos x = 0$ in the interval $\left(0, \frac{\pi}{2}\right)$	(p)	1
(B)	Value(s) of k for which the planes $kx + 4y + z = 0$, $4x + ky + 2z = 0$ and $2x + 2y + z = 0$ intersect in a straight line	(q)	2
(C)	Value(s) of k for which $\lvert x - 1\rvert + \lvert x - 2\rvert + \lvert x + 1\rvert + \lvert x + 2\rvert = 4k$ has integer solution(s)	(r)	3
(D)	If $y' = y + 1$ and $y(0) = 1$ then value(s) of y(ln2)	(s)	4
		(t)	5

Sol. **(A) → (p)** **(B) → (q, s)** **(C) → (q, r, s, t)** **(D) → (r)**

(A). $f'(x) > 0,\ \forall\ x \in (0, \pi/2)$

$f(0) < 0$ and $f(\pi/2) > 0$

so one solution.

(B). Let (a, b, c) is direction ratio of the intersected line, then

$ak + 4b + c = 0$

$4a + kb + 2c = 0$

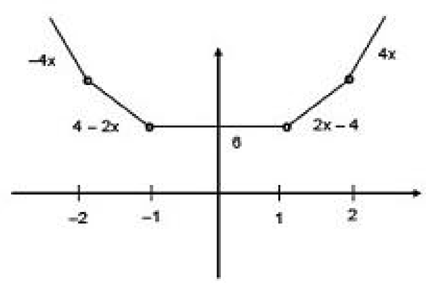
$$\frac{a}{8-k} = \frac{b}{4-2k} = \frac{c}{k^2-16}$$

We must have $2(8 - k) + 2(4 - 2k) + (k^2 - 16) = 0$

$\Rightarrow k = 2, 4.$

(C). Let $f(x) = |x + 2| + |x + 1| + |x - 1| + |x - 2|$

$\Rightarrow$ k can take value 2, 3, 4, 5.



(D). $\int \frac{dy}{y+1} = \int dx$

$\Rightarrow f(x) = 2e^x - 1$

$\Rightarrow f(\ln 2) = 3$

30. Match the statements/expressions in **Column I** with the values given in **Column II.**

	Column I		Column II
(A)	Root(s) of the expression $2\sin^2\theta + \sin^2 2\theta = 2$	(p)	$\frac{\pi}{6}$
(B)	Points of discontinuity of the function $f(x) = \left[\frac{6x}{\pi}\right]\cos\left[\frac{3x}{\pi}\right]$, where $[y]$ denotes the largest integer less than or equal to y	(q)	$\frac{\pi}{4}$
(C)	Volume of the parallelopiped with its edges represented by the vectors $\hat{i}+\hat{j}$, $\hat{i}+2\hat{j}$ and $\hat{i}+\hat{j}+\pi\hat{k}$	(r)	$\frac{\pi}{3}$
(D)	Angle between vectors $\vec{a}$ and $\vec{b}$ where $\vec{a}$, $\vec{b}$ and $\vec{c}$ are unit vectors satisfying $\vec{a}+\vec{b}+\sqrt{3}\vec{c} = \vec{0}$	(s)	$\frac{\pi}{2}$
		(t)	π

Sol. **(A) → (q, s) (B) → (p, r, s, t) (C) → (t) (D) → (r)**

(A). $2\sin^2\theta + 4\sin^2\theta\cos^2\theta = 2$

$\sin^2\theta + 2\sin^2\theta(1 - \sin^2\theta) = 1$

$3\sin^2\theta - 2\sin^4\theta - 1 = 0 \Rightarrow \sin\theta = \pm\frac{1}{\sqrt{2}}, \pm 1$

$\Rightarrow \theta = \frac{\pi}{4}, \frac{\pi}{2}$.

(B). Let $y = \frac{3x}{\pi}$

$\Rightarrow \frac{1}{2} \le y \le 3 \ \forall x \in \left[\frac{\pi}{6}, \pi\right]$

Now $f(y) = [2y]\cos[y]$

Critical points are $y = \frac{1}{2}, y = 1, y = \frac{3}{2}, y = 3$

$\Rightarrow$ points of discontinuity $\left\{\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \pi\right\}$.

(C). $\begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & \pi \end{vmatrix} = \pi \Rightarrow$ volume of parallelopiped $= \pi$

(D). $|\vec{a}+\vec{b}| = \sqrt{3}$

$\Rightarrow \sqrt{2 + 2\cos\alpha} = \sqrt{3}$

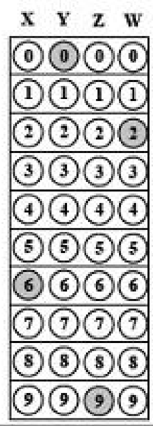
$\Rightarrow 2 + 2\cos\alpha = 3$

$\Rightarrow \alpha = \frac{\pi}{3}$.

SECTION – IV

Integer Answer Type

This section contains 8 questions. The answer to each of the questions is a single digit integer, ranging from 0 to 9. The appropriate bubbles below the respective question numbers in the ORS have to be darkened. For example, if the correct answers to question numbers X, Y, Z and W (say) are 6, 0, 9 and 2, respectively, then the correct darkening of bubbles will look like the following:



31. Let f: R → R be a continuous function which satisfies $f(x) = \int_0^x f(t)dt$. Then the value of f(ln5) is

Sol. **0**

$f(x) = \int_0^x f(t)dt \Rightarrow f(0) = 0$

also, $f'(x) = f(x), x > 0$

$\Rightarrow f(x) = ke^x, x > 0$

$\because$ $f(0) = 0$ and $f(x)$ is continuous $\Rightarrow f(x) = 0 \ \forall \ x > 0$

$\therefore$ $f(\ln 5) = 0$.

32. The centres of two circles C_1 and C_2 each of unit radius are at a distance of 6 units from each other. Let P be the mid point of the line segment joining the centres of C_1 and C_2 and C be a circle touching circles C_1 and C_2 externally. If a common tangent to C_1 and C passing through P is also a common tangent to C_2 and C, then the radius of the circle C is

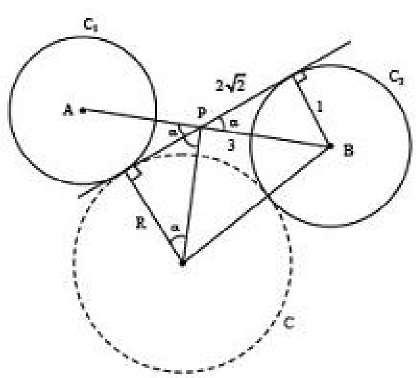
Sol. **8**

$\cos\alpha = \frac{2\sqrt{2}}{3}$

$\sin\alpha = \frac{1}{3}$

$\tan\alpha = \frac{2\sqrt{2}}{R}$

$\Rightarrow R = \frac{2\sqrt{2}}{\tan\alpha} = 8$ units.



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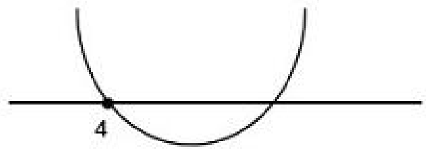
Alternate:

$(R+1)^2 = (R-1)^2 + (4\sqrt{2})^2$

$\Rightarrow R = 8.$

33. The smallest value of k, for which both the roots of the equation $x^2 - 8kx + 16(k^2 - k + 1) = 0$ are real, distinct and have values at least 4, is

Sol. 2



$x^2 - 8kx + 16(k^2 - k + 1) = 0$

$D > 0 \Rightarrow k > 1$ ... (1)

$\frac{-b}{2a} > 4 \Rightarrow \frac{8k}{2} > 4$

$\Rightarrow k > 1$ ... (2)

$f(4) \ge 0 \Rightarrow 16 - 32k + 16(k^2 - k + 1) \ge 0$

$k^2 - 3k + 2 \ge 0$

$k \le 1 \cup k \ge 2$ ... (3)

Using (1), (2) and (3)

$k_{min} = 2.$

34. The maximum value of the function $f(x) = 2x^3 - 15x^2 + 36x - 48$ on the set $A = \{x | x^2 + 20 \le 9x|\}$ is

Sol. 7

$f'(x) = 6(x-2)(x-3)$

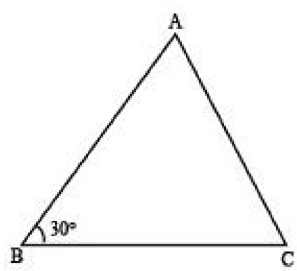
so $f(x)$ is increasing in $(3, \infty)$

Also, $A = \{4 \le x \le 5\}$

$\therefore f_{max} = f(5) = 7.$

35. Let ABC and ABC′ be two non–congruent triangles with sides AB = 4, AC = AC′ = $2\sqrt{2}$ and angle B = 30°. The absolute value of the difference between the areas of these triangles is

Sol. 4



$\cos\beta = \frac{a^2 + 16 - 8}{2 \times a \times 4}$

$\Rightarrow \frac{\sqrt{3}}{2} = \frac{a^2 + 8}{8a}$

$\Rightarrow a^2 - 4\sqrt{3}a + 8 = 0$

$\Rightarrow a_1 + a_2 = 4\sqrt{3}, \ a_1 a_2 = 8$

$\Rightarrow |a_1 - a_2| = 4$

$\Rightarrow |\Delta_1 - \Delta_2| = \frac{1}{2} \times 4 \sin 30° \times 4 = 4.$

36. If the function $f(x) = x^3 + e^{x/2}$ and $g(x) = f^{-1}(x)$, then the value of $g'(1)$ is

Sol. 2

$f(0) = 1, \ f'(x) = 3x^2 + \frac{1}{2}e^{x/2}$

$\Rightarrow f'(g(x))\, g'(x) = 1$

Put $x = 0 \Rightarrow g'(1) = \frac{1}{f'(0)} = 2.$

37. Let p(x) be a polynomial of degree 4 having extremum at x = 1, 2 and $\lim_{x \to 0}\left(1+\frac{p(x)}{x^2}\right) = 2$. Then the value of p(2) is

Sol. **0**

Let $P(x) = ax^4 + bx^3 + cx^2 + dx + e$

$P'(1) = P'(2) = 0$

$\lim_{x \to 0}\left(\frac{x^2 + P(x)}{x^2}\right) = 2$

$\Rightarrow P(0) = 0 \Rightarrow e = 0$

$\lim_{x \to 0}\left(\frac{2x + P'(x)}{2x}\right) = 2$

$\Rightarrow P'(0) = 0 \;\Rightarrow d = 0$

$\lim_{x \to 0}\left(\frac{2 + P''(x)}{2}\right) = 2$

$\Rightarrow c = 1$

On solving, a = 1/4, b = – 1

So $P(x) = \frac{x^4}{4} - x^3 + x^2$

$\Rightarrow P(2) = 0.$

38. Let (x, y, z) be points with integer coordinates satisfying the system of homogeneous equations:

$3x - y - z = 0$

$-3x + z = 0$

$-3x + 2y + z = 0.$

Then the number of such points for which $x^2 + y^2 + z^2 \le 100$ is

Sol. 7

$3x - y - z = 0$

$-3x + z = 0$

$\Rightarrow y = 0$

and $z = 3x$

$\Rightarrow x^2 + y^2 + z^2 = x^2 + z^2 = x^2 + 9x^2 = 10x^2 \le 100$

$\Rightarrow x^2 \le 10$

$\Rightarrow x = 0, \pm 1, \pm 2, \pm 3$

There are such seven points.