Download IIT-JEE Previous Year Paper - II “2011” (Math)

44. A value of b for which the equations

$x^2 + bx - 1 = 0$

$x^2 + x + b = 0$

have one root in common is

(a) $-\sqrt{2}$ (b) $-i\sqrt{3}$ (c) $i\sqrt{5}$ (d) $\sqrt{2}$

Ans : **(b)** α be the common root

$\alpha^2 + b\alpha - 1 = 0$

$\alpha^2 + \alpha + b = 0$

$\Rightarrow \alpha^2 = \dfrac{b^2+1}{1-b},\ \alpha = \dfrac{1+b}{b-1}\ \therefore\ b^3 + 3b = 0 \Rightarrow b^2 = -3\ \therefore b = -i\sqrt{3}$

45. Let $\omega \neq 1$ be a cube root of unity and S be the set of all non – singular matrices of the form $\begin{bmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{bmatrix}$, where each of a, b and c is either ω or ω^2. Then the number of distinct matrices in the set S is

(a) 2 (b) 6 (c) 4 (d) 8

Ans : **(a)** $\begin{vmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{vmatrix} \neq 0$

$\Rightarrow 1 - (a+c)\omega + ac\omega^2 \neq 0\ \therefore\ a + c \neq -1\quad ac \neq 1$

Clearly a = ω or ω^2 or c = ω or ω^2

but $c \neq \omega^2$ as it makes the determinant zero.

also $a \neq \omega^2$

$\therefore a = \omega, b = \omega^2$ or $\omega, c = \omega$

$\therefore$ No of such matrices = 2

46. The circle passing through the point (-1, 0) and touching the y – axis at (0, 2) also passes through the point

(a) (-3/2, 0) (b) (-5/2,2) (c) (-3/2, 5/2) (d) (-4, 0)

Ans : **(d)** Let C be (h, 2)

Equation of circle:

$\because$ circle passes through (-1, 0) , h = -5/2

Equation of circle will be $(x + (5/2))^2 + (y - 2)^2 = 25/4$

Only (-4, 0) satisfies

47. If $\lim_{x\to 0}\left[1 + x\ln(1+b^2)\right]^{1/x} = 2b\sin^2\theta$, $b > 0$ and $\theta \in (-\pi, \pi]$ then the value of θ is

(a) $\pm\pi/4$ (b) $\pm\pi/3$ (c) $\pm\pi/6$ (d) $\pm\pi/2$

Ans : **(d)** $1 + b^2 = 2b\sin^2\theta \Rightarrow b^2 - 2b\sin^2\theta + 1 = 0$

$\Rightarrow b = \dfrac{2\sin^2\theta \pm \sqrt{4\sin^4\theta - 4}}{2} = \sin^2\theta \pm \sqrt{-\cos^2\theta(1+\sin^2\theta)}$

For b to be real, $\cos\theta = 0 \Rightarrow \theta = \pm\pi/2$

48. Let f: $[-1,2]\to(0,\infty]$ be a continuous function such that f(x) = f(1 –x) for all x $\in[-1,2]$. Let $R_1=\int_{-1}^{2}xf(x)dx$, and R_2 be the area of the region bounded by y = f(x), x = -1, x = 2, and the x – axis. Then

(a) $R_1=2R_2$ (b) $R_1=3R_2$ (c) $2R_1=R_2$ (d) $3R_1=R_2$

Ans : **(c)** $R_1=\int_{-1}^{2}(1-x)f(1-x)dx=R_2-R_1\Rightarrow 2R_1=R_2$

SECTION – II (TOTAL MARKS: 16)
MULTIPLE CORRECT CHOICE TYPE

This section contains 4 multiple choice questions. Each question has four choices (a), (b), (c) and (d) out of which **ONE or MORE** may be correct.

49. Let L be a normal to the parabola $y^2=4x$. If L passes through the point (9, 6), the L is given by

(a) $y-x+3=0$ (b) $y+3x-33=0$
(c) $y+x-15$ (d) $y-2x+12=0$

Ans : **(a,b,d)** Equation of normal $y=mx-2am-am^3$ i.e $y=mx-2m-m^3$

Passing through (9, 6) $\therefore m^3-7m+6=0\Rightarrow m=-3,2,1$

$\therefore$ Equation of normals are $y+3x-33=0$, $y-2x+12=0$, $y-x+3=0$

50. Let f: $(0,1)\to R$ be defined by $f(x)=\frac{b-x}{1-bx}$, where b is a constant such that 0 < b < 1. Then

(a) f is not invertible on (0, 1) (b) $f\neq f^{-1}$ on (0,1) and $f'(b)=\frac{1}{f'(0)}$

(c) $f=f^{-1}$ on (0,1) and $f'(b)=\frac{1}{f'(0)}$ (d) f^{-1} is differentiable on (0,1)

Ans : **(c, d)** $f'(x)=\frac{(b^2-1)}{(1-bx)^2}<0\ \forall\ x\in(0,1)$ as $b\in(0,1)$

Also, f(x) is continuous for all $x\in(0,1)$ as domain of definition is $R-\{1/b\}$ where 1/b > 1. Hence, f(x) is strictly decreasing, so invertible.

$f(x)=y=\frac{b-x}{1-bx}\Rightarrow x=\frac{y-b}{yb-1}\Rightarrow f^{-1}(x)=\frac{b-x}{1-bx}=f(x)$

$f'(b)=\frac{1}{b^2-1}$ and $f'(0)=b^2-1$

Since , $f'(x)$ exist for all $x\in(0,1)\Rightarrow f^{-1}$ is differentiable for all $x\in(0,1)$

51. Let E and F be two independent events. The probability that exactly one of them occurs is 11/25 and the probability of none of them occurring is 2/25. If P(T) denotes the probability of occurrence of the event T, then

(a) $P(E,) = 4/5$, $P(F) = 3/5$ (b) $P(E) = 1/5, P(F) = 2/5$

(c) $P(E) = 2/5$, $P(F) = 1/5$ (d) $P(E) = 3/5, P(F) = 4/5$

Ans : **(a, d)** $P(E \cap \bar{F}) + P(\bar{E} \cap F) = 11/25$, $P(\bar{E} \cap \bar{F}) = 2/25$

Let P(E) = x, P(F) = y $\therefore P(\bar{E}) \cdot P(\bar{F}) = 2/25 \Rightarrow (1-x)(1-y) = 2/25$

Also $P(E) \cdot P(\bar{F}) + P(\bar{E}) \cdot P(F) = 11/25 \Rightarrow x(1-y) + (1-x)y = 11/25$

Solving x = 3/5 or 4/5 and y = 4/5 or 3/5

52. If $f(x) = \begin{cases} -x - \frac{\pi}{2}, x \le -\frac{\pi}{2} \\ -\cos x, \ -\frac{\pi}{2} < x \le 0, \text{then} \\ x-1, \ 0 < x \le 1 \\ \ln x, \ x > 1 \end{cases}$

(a) f(x) is continuous at $x = \pi/2$ (b) f(x) is not differentiable at x = 0

(c) f(x) is differentiable at x = 1 (d) f(x) is differentiable x = -3/2

Ans : **(abcd)** $f(x) = \begin{cases} -x - \frac{\pi}{2}, x \le -\frac{\pi}{2} \\ -\cos x, \ -\frac{\pi}{2} < x \le 0, \\ x-1, \ 0 < x \le 1 \\ \ln x, \ x > 1 \end{cases}$

Continuity at $x = -\pi/2$

$\text{L.H.L} = \text{R.H.L} = f(-\pi/2) = 0$

$\therefore f'(x) = \begin{cases} -1, \ x \le -\frac{\pi}{2} \\ \sin x, \ -\frac{\pi}{2} < x \le 0 \\ 1, \ 0 < x \le 1 \\ 1/x, \ x > 1 \end{cases}$

Clearly, f(x) is not differentiable at x = 0, f(x) is differentiable at x = 1 and f(x) is differentiable x = -3/2

SECTION – III (TOTAL MARKS: 24)
(INTEGER ANSWER TYPE)

This section contains **6 questions**. The answer to each of the questions is a **single – digit integer**, ranging from 0 to 9. The bubble corresponding to the correct answer is to be darkened in the ORS.

53. Let $\omega = e^{i\pi/3}$, and a, b, c, x, y, z be non – zero complex numbers such that

$$a + b + c = X$$
$$a + b\omega + c\omega^2 = y$$
$$a + b\omega^2 + c\omega = z$$

Then the value of $\dfrac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2}$ is

Ans : **3** $|x|^2 + |y|^2 + |z|^2 = xx^c + yy^c + zz^c$

$= (a + b + c)$

$(a^c + b^c + c^c) + (a + b\omega + c\omega^2)(a^c + b^c\omega^2 + c^c\omega) + (a + b\omega^2 + c\omega)(a^c + b^c\omega + c^c\omega^2)$

$= 3(|a|^2 + |b|^2 + |c|^2) \Rightarrow \dfrac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2} = 3$

54. Let $y'(x) + y(x)g'(x) = g(x)g'(x)$, $y(0) = 0$, $x \in R$, where f'(x) denotes $\dfrac{df(x)}{dx}$ and g(x) is a given non – constant differentiable function on R with g(0) = g(2) = 0. Then the value of y(2) is

Ans : **0** $y'(x) + y(x)g'(x) = g(x)g'(x)$

Linear differential equation with integrating factor $e^{g(x)}$

$\Rightarrow y(x)\cdot e^{g(x)} = \int g(x)\cdot g'(x)\cdot e^{g(x)}dx \Rightarrow y(x)\cdot e^{g(x)} = e^{g(x)}(g(x)-1)+c$

Since, $y(0) = 0$ and $g(0) = 0 \Rightarrow c = 1$

$\Rightarrow y(x) = (g(x)-1) + e^{-g(x)} \Rightarrow y(2) = (g(2)-1) + e^{-g(2)} = 0$

55. Let M be a 3×3 matrix satisfying

$$M\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}-1\\2\\3\end{bmatrix},\quad M\begin{bmatrix}1\\-1\\0\end{bmatrix} = \begin{bmatrix}1\\1\\-1\end{bmatrix} \text{ and } M\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1\\1\\12\end{bmatrix}$$

Then the sum of the diagonal entries of M is

Ans : **9** $(\vec{r}-\vec{c})\times\vec{b}=0 \Rightarrow \vec{r}=\vec{c}+\lambda\vec{b}=(1-\lambda)\hat{j}+(2+\lambda)\hat{j}+3\hat{k}$

If $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$, then $\vec{r}.\vec{a}=0 \Rightarrow -x-z=0 \Rightarrow x=-z$

Also, $x=1-\lambda,\ y=2+\lambda,\ z=3 \Rightarrow -3=1-\lambda \Rightarrow \lambda=4$

$\therefore \vec{r}=-3\hat{j}+6\hat{j}+3\hat{k}$ $\quad\quad \therefore \vec{r}\cdot\vec{b}=3+6=9$

58. The number of distinct real roots of $x^4-4x^3+12x^2+x-1=0$ is

Ans : **2** Let us assume that all four roots are real and distinct

Hence, f'(x) = 0 must have 3 distinct real roots and

$f''(x)=0$ must have 2 distinct real roots, but that is not true as

$f''(x)=12(x^2-2x+2)$ with $D<0$

Hence, f(x) = 0 can't have all four roots real

As, f(0) = -1, f(1) =9 and f(-1) = 15

$\Rightarrow f(x)=0$ must have two distinct roots one in (-1, 0) and the other one in (0, 1)

SECTION – II (TOTAL MARKS: 16)
(MULTIPLE CORRECT ANSWERS TYPE)

This section contains **2 questions**. Each question has four statements (A, B, C and D) given in **Column I** and five statements (p, q , r, s and t) in **Column II**. Any given statement in **Column I** can have correct matching with **ONE and MORE** statement(s) given in **Column II**. For example, if for a given question, Statement B matches with the statements given in q and r, then for the particular question, against statement B, darken the bubbles corresponding to q and r in the ORS.

59. Match the statements given in Column I with the values given in Column II

	Column I		**Column II**
(A)	If $\vec{a} = \hat{j} + \sqrt{3}\hat{k}, \vec{b} = -\hat{j} + \sqrt{3}\hat{k}$ and $\vec{c} = 2\sqrt{3}\hat{k}$ from a triangle, then the internal angle of the triangle between $\vec{a}$ and $\vec{b}$ be	(p)	$\frac{\pi}{6}$
(B)	If $\int_a^b (f(x) - 3x)dx = a^2 - b^2$, then the value of $f\left(\frac{\pi}{6}\right)$ is	(q)	$\frac{2\pi}{3}$
(C)	The value of $\frac{\pi^2}{\ln 3}\int_{7/6}^{5/6} \sec(\pi x)dx$ is	(r)	$\frac{\pi}{3}$
(D)	The maximum value of $\left|\text{Arg}\left(\frac{1}{1-z}\right)\right|$ for $|z| = 1,\ z \neq 1$ is given by	(r)	π
		(t)	$\frac{\pi}{2}$

Ans : $(A) \to (q), (B) \to (p), (C) \to (s), (D) \to (t)$

For (A): Since, side lengths are 2, 2, $2\sqrt{3}$, hence angle between $\vec{a}$ and $\vec{b}$ is

$$\cos\theta = \frac{4+4-12}{2\cdot 2\cdot 2} = -1/2 \Rightarrow \theta = 2\pi/3$$

For (B) : $\int_a^b f(x)dx = \frac{1}{2}(b^2 - a^2) \Rightarrow f(x) = x \Rightarrow f(\pi/6) = \pi/6$

For (C) : $\frac{\pi}{\ln 3}\left((\ln\sec\pi x + \tan\pi x)\right)_{7/6}^{5/6} = \frac{\pi}{\ln 3}\left[\ln\sqrt{3} - \ln\frac{1}{\sqrt{3}}\right] = \pi$

For (D) : $z = e^{i\theta}$, $\frac{1}{1-z} = \frac{(1-\cos\theta) + i\sin\theta}{(1-\cos\theta)^2 + \sin^2\theta}$

$\therefore \arg\left(\frac{1}{1-z}\right) = \frac{\sin\theta}{1-\cos\theta} = f(\theta)$, which is maximum when $\theta = \pi/2$

60. Match the statements given in Column I with the intervals/ union of intervals given in Column II

	Column I		Column II
(A)	The set $\left\{\text{Re}\left(\frac{2iz}{1-z^2}\right): z \text{ is complex number, } \lvert z\rvert = 1,\ z \neq \pm 1\right\}$	(p)	$(-\infty,-1)\cup(1,\infty)$
(B)	The domain of the function $f(x)=\sin^{-1}\left(\frac{8(3)^{x-2}}{1-3^{2(x-1)}}\right)$ is	(q)	$(-\infty,0)\cup(0,\infty)$
(C)	If $f(\theta)=\begin{vmatrix} 1 & \tan\theta & 1 \\ -\tan\theta & 1 & \tan\theta \\ -1 & -\tan\theta & 1 \end{vmatrix}$, then the set $\left\{f(\theta): 0\le\theta<\frac{\pi}{2}\right\}$ is	(r)	$[2,\infty)$
(D)	If $f(x)=x^{3/2}(3x-10), x\ge 0$, then f(x) is increasing in	(s)	$(-\infty,-1]\cup[1,\infty)$
		(t)	$(-\infty,0]\cup[2,\infty)$

Ans : $(A)\to(s)$ Let $k = \text{Re}\left(\frac{2iz}{1-z^2}\right) = \text{Re}\left(\frac{2ie^{i\theta}}{1-e^{2i\theta}}\right) = \text{Re}\left(\frac{2i(\cos\theta+i\sin\theta)}{2\sin^2\theta-2i\sin\theta\cos\theta}\right)$

$$= \text{Re}\left(\frac{-2i(\cos\theta+i\sin\theta)}{2i\sin\theta(\cos\theta+i\sin\theta)}\right) = \frac{-1}{\sin\theta}$$

It is defined only when $k \in (-\infty,-1]\cup[1,\infty)$

$(B)\to(t)$ Since, $-1 \le \frac{8\cdot(3^{x-2})}{1-3^{2x-2}} \le 1$

$\Rightarrow -1 \le \frac{8.3^x}{9-3^{2x}} \le 1$

Put $y = 3^x$

$\Rightarrow -1 \le \frac{8y}{9-y^2} \le 1$

$\therefore -1 \le \frac{8y}{9-y^2} \Rightarrow \frac{(y-9)(y+1)}{(y+3)(y-3)} \ge 0$

$\Rightarrow \frac{(y-9)}{(y-3)} \ge 0$ ($\because$ y + 1 and y + 3 are always (+) ve)

$\Rightarrow y < 3$ and $y \ge 9$

$\Rightarrow x \in (-\infty,1)\cup[2,\infty)$

and $\frac{8y}{9-y^2} \le 1 \Rightarrow \frac{(y+9)(y-1)}{(y+3)(y-3)} \ge 0 \Rightarrow \frac{(y-1)}{(y-3)} \ge 0$

$\Rightarrow y \le 1$ and $y > 3$

$\Rightarrow x \in (-\infty, 0] \cup (2, \infty)$

$\therefore$ The common solution is $x \in (-\infty, 0] \cup (2, \infty)$.

$(C) \rightarrow (r)$

Since, $f(\theta) = \begin{vmatrix} 1 & \tan\theta & 1 \\ -\tan\theta & 1 & \tan\theta \\ -1 & -\tan\theta & 1 \end{vmatrix} = 2\sec^2\theta \therefore f(\theta) \in [2, \infty)$

$(D) \rightarrow (r)$ Let $f(x) = x^{3/2}(3x - 10)$

$\Rightarrow f'(x) = (3/2)x^{1/2}(3x - 10) + x^{3/2} \times 3$

For increasing,

$\therefore f'(x) \geq 0$

$\Rightarrow 3\sqrt{x}\left[(3x/2) - 5 + x\right] \geq 0$

$\Rightarrow 3\sqrt{x} \cdot (5x - 10) \geq 0 \Rightarrow x \geq 2$